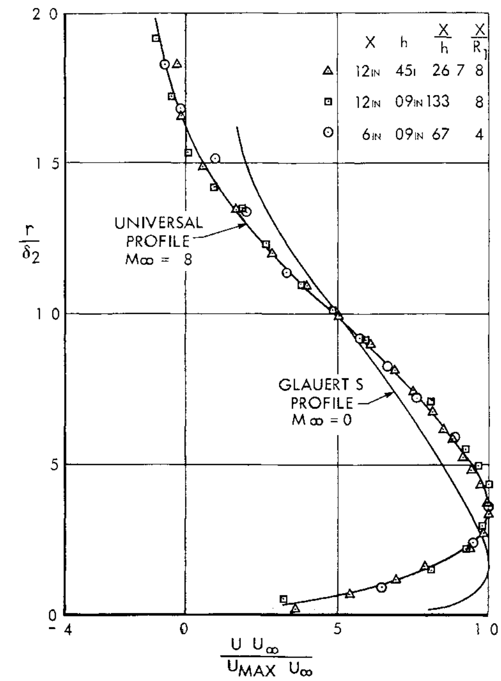

Fig 4 Axisymmetric compressible wall-jet velocity profiles; M_∞ = 0 8, $(Pt/P)_F$ = 4 4, convergent nozzles

ordinate allows the data to be represented by a single universal profile; Glauert's profile for $M_\infty = 0$ is compared, and it differs considerably

Figure 5 is a case where $U_{max} \rightarrow U_\infty$; here the skin friction predominates, and none of the profiles fall on the universal profile for M_∞ = 0 8 (from Fig 4) Profiles at other values of $(Pt/P)_F$, X/h, and X/R_1 were plotted (not shown) and compared with the profile of Fig 4 Figure 6 was constructed to show the approximate region of applicability of the universal profile of Fig 4 The curve $(X/h)_{co\ e}$ vs $(Pt/P)_F$

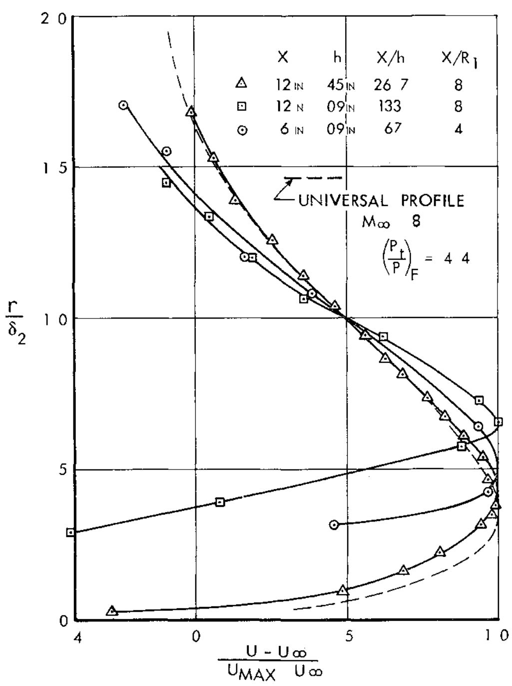

Fig 5 Axisymmetric compressible wall-jet velocity profiles; M_∞ = 0 8, $(Pt/P)_F$ = 2 0, convergent nozzles

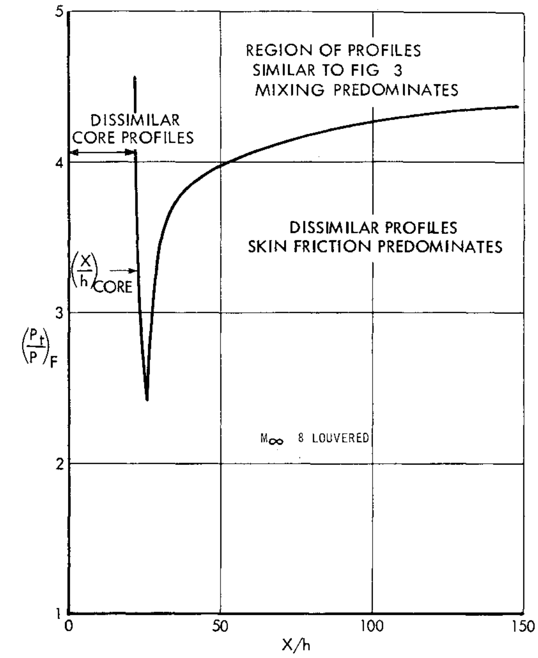

Fig 6 Axisymmetric compressible wall jet: region of profile similarity; M_∞ = 0 8, convergent nozzles

was estimated from data taken during the experiment; it represents the expected length of the core region

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Wave Propagation in Rotating Elastic Media

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The effect of "rigid-body" rotation on wave propagation velocities in elastic media is investigated It is found that the rotation generates a coupling between the classical longitudinal and transverse waves The rotation tends to increase the propagation velocity of transverse-type amplitude waves while decreasing the propagation velocity of longitudinal-type amplitude waves For phase waves, which are associated with vibratory motion, the situation is found to be reversed

Introduction

IT is a well-known result that, in an unbounded homogeneous isotropic elastic medium, disturbances are propagated as longitudinal or transverse waves with velocities $[(\lambda + 2\mu)/\rho]^{1/2}$ and $[\mu/\rho]^{1/2}$, respectively, where λ and μ are Lame's elastic constants and ρ is the mass density of the medium The purpose here is to investigate the effect of "rigid body" rotation of the elastic medium on these propagation velocities

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Governing Equations

The governing dynamical equations may be written in the form[1]

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} = \rho a_i \quad (1)$$

where u_i and a_i are, respectively, the displacement and acceleration of a typical point of the elastic medium The notation employed is the conventional Cartesian tensor notation, with the comma denoting partial differentiation with respect to the space variables indicated by the subsequent subscripts The acceleration must be computed in a Newtonian reference frame and may be expressed as

$$a_i = (\partial^2 u_i/\partial t^2) + e_{ijk}e_{klm}\Omega_j\Omega_l x_m + 2e_{ijk}\Omega_j(\partial u_k/\partial t) \quad (2)$$

where the Cartesian coordinate system x_i is considered to be rotating with uniform angular velocity Ω_i The partial time derivatives are computed with respect to the rotating system, and e_{ijk} is the Cartesian alternating tensor The development of Eq (2) follows from rewriting in tensor form the general expression for acceleration found in Kane[2] and using the assumptions of uniform rotation and small displacement

Substituting Eq (2) into Eq (1) leads to the equations

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} - \rho(\partial^2 u_i/\partial t^2) - 2\rho e_{ijk}\Omega_j(\partial u_k/\partial t) = \rho e_{ijk}e_{klm}\Omega_j\Omega_l x_m \quad (3)$$

Hence, the homogeneous equations of motion are

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} - \rho(\partial^2 u_i/\partial t^2) - 2\rho e_{ijk}\Omega_j(\partial u_k/\partial t) = 0 \quad (4)$$

Wave Propagation

The technique used for the investigation of wave propagation is that of Synge[3] in his work on the motion of viscous fluids conducting heat Following this technique, solutions of Eq (4) are taken in the form

$$u_i = A_i \exp(a_k x_k + bt) \quad (5)$$

where the A_i are constants, and a_i and b are complex constants that may be expressed as

$$a_k = a_k' + ia_k'' \qquad b = b' + ib'' \quad (6)$$

where the primed quantities are real constants The equations

$$a_k'x_k + b't = \text{const} \qquad a_k''x_k + b''t = \text{const} \quad (7)$$

represent, respectively, "amplitude" waves and "phase" waves or vibrating motion The squares of the respective propagation velocities are given by

$$(V')^2 = (b')^2/(a_k'a_k')^2 \qquad (V'')^2 = (b'')^2/(a_k''a_k'')^2 \quad (8)$$

Substituting Eq (5) into Eq (4) leads to

$$(\lambda + \mu)A_k a_k a_i + (\mu a^2 - \rho b^2)A_i - 2\rho b e_{ijk}\Omega_j A_k = 0 \quad (9)$$

where

$$a^2 = a_k a_k \quad (10)$$

Consider now waves propagated in planes normal to the axis of rotation If this axis is taken to be x_3, a_3 and Ω_i are given by

$$a_3 = \Omega_1 = \Omega_2 = 0 \qquad \Omega_3 = \Omega \quad (11)$$

The determinantal equation then becomes

$$\begin{vmatrix} [\mu a^2 - \rho b^2 + (\lambda + \mu)a_1^2] & [(\lambda + \mu)a_1a_2 + 2\rho b\Omega] \\ [(\lambda + \mu)a_1a_2 - 2b\rho\Omega] & [\mu a^2 - \rho b^2 + (\lambda + \mu)a_2^2] \end{vmatrix} = 0$$

or

$$(b^2/a^2)^2 - (b^2/a^2)[(\mu/\rho) + \{(\lambda + 2\mu)/\rho\} - (4\Omega^2/a^2)] + (\mu/\rho)(\lambda + 2\mu)/\rho = 0 \quad (12)$$

Finally, solving for b^2/a^2 leads to the expression

$$b^2/a^2 = \tfrac{1}{2}\{[\mu/\rho] + [(\lambda + 2\mu)/\rho] - (4\Omega^2/a^2)\} \pm \{\tfrac{1}{4}([(\lambda + 2\mu)/\rho] - [\mu/\rho])^2 - (2\Omega^2/a^2)([(\lambda + 2\mu)/\rho] + [\mu/\rho]) + (4\Omega^4/a^4)\}^{1/2} \quad (13)$$

Equation (13) shows that the rotation introduces a coupling between the transverse and longitudinal waves The nature of this coupling is revealed more explicitly, however, by considering Ω^2/a^2 small as compared with μ/ρ and by expanding the radical This leads to the equations

$$b^2/a^2 \approx [(\lambda + 2\mu)/\rho] - 4[(\lambda + 2\mu)/(\lambda + \mu)](\Omega^2/a^2) \quad (14)$$

and

$$b^2/a^2 \approx [\mu/\rho] + 4[(\lambda + 2\mu)/(\lambda + \mu)](\Omega^2/a^2) \quad (15)$$

These equations show that the propagation is still of a longitudinal and transverse nature, but for a_i,b real (amplitude waves), the rotation tends to increase the velocity of the transverse-type waves while decreasing the velocity of the longitudinal type waves For a_i,b imaginary (phase waves), the situation is reversed This latter case is an indication of the effect of rotation on the natural longitudinal and transverse vibration frequencies in rotating elastic bodies

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Derivation of Element Stiffness Matrices

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Recent derivation of element stiffness matrices of axisymmetrical shells by Grafton and Strome[1] is an example of the direct determination of stiffness matrices through the principle of virtual displacement[2,3] In applying this method, the displacement of a structural element is first expressed in terms of n undetermined coefficients, where n is the same as the number of the generalized displacement of the structural elements In general, for such analysis, the corresponding stress distribution of the element will not satisfy the equations of equilibrium The present note is to show that the displacement function may be assumed to contain more than n undeterminate coefficients, and the employment of the principle of minimum potential energy enables the evaluation of these additional coefficients Solutions obtained by taking more terms in the displacement function will represent an improvement in the equilibrium conditions

One begins by considering a displacement vector $\{u\}(=\{u, v, w\})$ that contains $(n + l)$ undeterminate coefficients $\{\alpha\}$ as follows:

$$\{u\} = [A] \underset{(n+l)\times 1}{\{\alpha\}} \quad (1)$$

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